

The Real Story Behind Story Problems: Effects of Representations on Quantitative Reasoning

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This article explores how differences in problem representations change both the performance and underlying cognitive processes of beginning algebra students engaged in quantitative reasoning. Contrary to beliefs held by practitioners and researchers in mathematics education, students were more successful solving simple algebra story problems than solving mathematically equivalent equations. Contrary to some views of situated cognition, this result is not simply a consequence of situated world knowledge facilitating problem-solving performance, but rather a consequence of student difficulties with comprehending the formal symbolic representation of quantitative relations. We draw on analyses of students' strategies and errors as the basis for a cognitive process explanation of when, why, and how differences in problem representation affect problem solving. We conclude that differences in external representations can affect performance and learning when one representation is easier to comprehend than another or when one representation elicits more reliable and meaningful solution strategies than another.

A commonly held belief about story problems at both the arithmetic and algebra levels is that they are notoriously difficult for students. Support for this belief can be seen among a variety of populations including the general public, textbook au-

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thors, teachers, mathematics education researchers, and learning science researchers. For evidence that this belief is commonly held within the general public, ask your neighbor. He or she is likely to express a sentiment along the lines of Gary Larson's cartoon captioned "Hell's Library" that contains bookshelves full of titles such as "Story Problems," "More Story Problems," and "Story Problems Galore." That many textbook authors believe in the greater difficulty of story problems is supported by an analysis of textbooks by Nathan, Long, and Alhali (2002). In 9 of the 10 textbooks they analyzed, new topics are initially presented through symbolic activities and only later are story problems presented, often as challenge problems. This ordering is consistent with the belief that symbolic representations are more accessible to students than are story problems.

More direct evidence of the common belief in the difficulty of story problems comes from surveys of teachers and mathematics educators. In a survey of 67 high school mathematics teachers, Nathan and Koedinger (2000a) found that most predicted that story problems would be harder than matched equations for algebra students. Nathan and Koedinger (2000a) also surveyed 35 mathematics education researchers, the majority of which also predicted that story problems would be harder than matched equations for algebra students. In another study of 105 K-12 mathematics teachers, Nathan and Koedinger (2000b) found that significantly more teachers agree than disagree with statements such as "Solving math problems presented in words should be taught only after students master solving the same problems presented as equations." This pattern was particularly strong among the high school teachers in the sample ($n = 30$).

Belief in the difficulty of story problems is also reflected in the learning science literature. Research on story problem solving, at both the arithmetic (Carpenter, Kepner, Corbit, Lindquist, & Reys, 1980; Cummins, Kintsch, Reusser, & Weimer, 1988; Kintsch & Greeno, 1985) and algebra levels (Clement, 1982; Nathan, Kintsch, & Young, 1992; Paige & Simon, 1966), has emphasized the difficulty of such problems. For instance, Cummins and colleagues (1988, p. 405) commented that "word problems are notoriously difficult to solve." They investigated first graders' performance on matched problems in story and numeric format for 18 categories of one-operator arithmetic problems. Students were 27% correct on the Compare 2 problem in story format ("Mary has 6 marbles. John has 2 marbles. How many marbles does John have less than Mary?") but were 100% correct on the matched numeric format problem $6 - 2 = ?$. Cummins et al. found performance on story problems was worse than performance on matched problems in numeric format for 14 of the 18 categories and was equivalent for the remaining 4 categories. Belief in the greater difficulty of story problems is also evident in the broader developmental literature. For instance, Geary (1994, p. 96) states that "children make more errors when solving word problems than when solving comparable number problems."

Although their research addressed elementary-level arithmetic problem solving, Cummins et al. (1988, p. 405) went on to make the broader claim that "as stu-

dents advance to more sophisticated domains, they continue to find word problems in those domains more difficult to solve than problems presented in symbolic format (e.g., algebraic equations)." However, apart from our own studies reported here, this broader claim appears to remain untested (cf. Reed, 1998). We have not found prior experimental comparisons of solution correctness on matched algebra story problems and equations for students learning algebra. In a related study, Mayer (1982a) used solution times to make inferences about the different strategies that well-prepared college students use on algebra word and equation problems. He found a different profile of solution times for word problems than for equations as problems varied in complexity and accounted for these differences by the hypothesis that students use a goal-based "isolate" strategy on equations and a less memory-intensive "reduce" strategy on word problems. Overall, students took significantly longer to solve one- to five-step word problems (about 15 sec) than matched equations (about 5 sec) with no reliable difference in number of errors (7% for word problems, 4% for equations). Whereas Mayer's study focused on timing differences for well-practiced participants, the studies reported here focus on error differences for beginning algebra students.

WHY ARE STORY PROBLEMS DIFFICULT?

What might account for the purported and observed difficulties of story problems? As many researchers have observed (Cummins et al., 1988; Hall et al., 1989; Lewis & Mayer, 1987; Mayer, 1982b), the process of story problem solving can be divided into a comprehension phase and a solution phase (see Figure 1). In the *comprehension phase*, problem solvers process the text of the story problem and create corresponding internal representations of the quantitative and situation-based relationships expressed in that text (Nathan et al., 1992). In the *solution phase* problem solvers use or transform the quantitative relationships that are represented both internally and externally to arrive at a solution. Two kinds of process explanations for the difficulty of story problems correspond with these two problem-solving phases. We will return to these explanations after describing how these two phases interact during problem solving (for more detail see Koedinger & MacLaren, 2002).

The comprehension and solution phases typically are interleaved rather than performed sequentially. Problem solvers iteratively comprehend first a small piece of the problem statement (e.g., a clause or sentence) and then produce a piece of corresponding external representation (e.g., an arithmetic operation or algebraic expression), often as an external memory aid. In Figure 1, the double-headed arrows within the larger arrows are intended to communicate this interactivity. During problem solving, aspects of newly constructed internal or external representations may influence further comprehension in later cycles (Kintsch, 1998). For example, after determining that the unknown value is the number of donuts, the

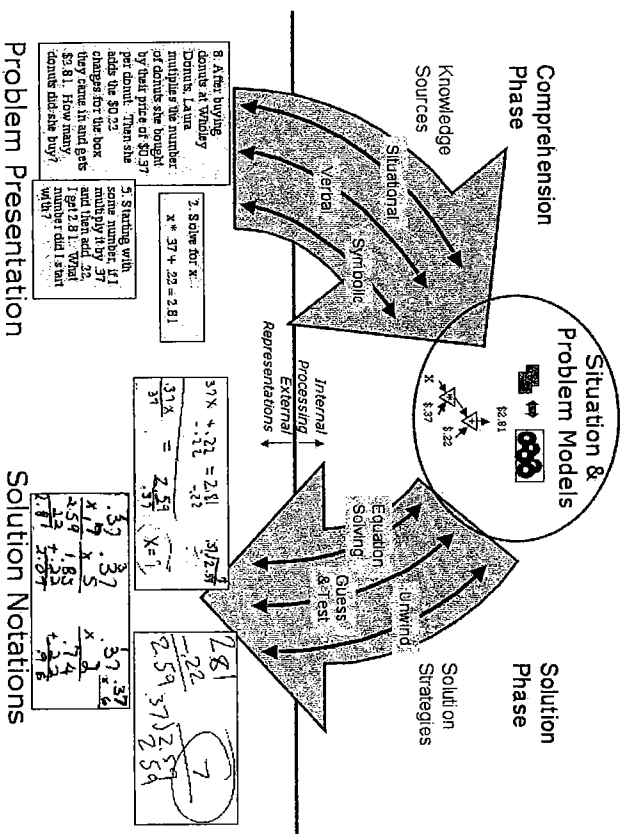


FIGURE 1 Quantitative problem solving involves two phases, comprehension and solution, both of which are influenced by the external representation (e.g., story, word, equation) in which a problem is presented. The influence on the comprehension phase results from the need for different kinds of linguistic processing knowledge (e.g., situational, verbal, or symbolic) required by different external representations. The impact on the solution phase results from the different computational characteristics of the strategies (e.g., unwind, guess-and-test, equation solving) cued by different external representations.

reader may then search for and reread a clause that uses number of donuts in a quantitative relation. Similarly, the production of aspects of the external representation may help maintain internal problem-solving goals that, in turn, may direct further comprehension processes.

A number of researchers have provided convincing evidence that errors in the comprehension phase well account for story problem solving difficulties (e.g., Cummins et al., 1988; Lewis & Mayer, 1987). For instance, Cummins et al. (1988) demonstrated that variations in first graders' story problem performance were well predicted by variations in problem recall and that both types of variation could be accounted for by difficulties students had in comprehending specific linguistic forms such as *some*, *more* *Xs than Ys*, and *altogether*. They concluded that "text comprehension factors figure heavily in word problem difficulty" (p. 435). Lewis and Mayer

(1987) summarized past studies with K–6 graders and their own studies with college students that showed more solution errors on arithmetic story problems with "inconsistent language" (e.g., when the problem says "more than" but subtraction is required to solve it) than problems with consistent language. Teachers' intuitions about the difficulty of algebra story problems (cf. Nathan & Koedinger, 2000a, 2000b) appear to be in line with these investigations of comprehension difficulties with arithmetic story problems. As one teacher explained, "Students are used to expressions written algebraically and have typically had the most practice with these ... translating 'English' or 'non-mathematical' words is a difficult task for many students."

A second process explanation for the difficulty of story problems focuses on the solution phase, particularly on the strategies students use to process aspects of the problem. A common view of how story problems are or should be solved, particularly at the algebra level, is that the problem text first is translated into written symbolic form and then the symbolic problem is solved (e.g., see Figure 2a). If problem solvers use this translate-and-solve strategy, then clearly story problems will be harder than matched symbolic problems because solving the written symbolic problem is an intermediate step in this case.

At the algebra level, the translate-and-solve strategy has a long tradition as the recommended approach. Regarding an algebra-level story problem, Paige and Simon (1966) comment, "At a common-sense level, it seems plausible that a person solves such problems by, first, translating the problem sentences into algebraic equations and, second, solving the equations." They go on to quote a 1929 textbook recommending this approach (Hawkes, Luby, & Touton, 1929). Modern textbooks also recommend this approach and typically present story problems as challenge problems and applications in the back of problem-solving sections (Nathan, Long, et al., 2002). Thus, a plausible source for teachers' belief in the difficulty of story problems over equations is the idea that equations are needed to solve story problems. An algebra teacher performing the problem difficulty ranking task described in Nathan and Koedinger (2000a) made the following reference to the translate-and-solve strategy (the numbers 1–6 refer to sample problems teachers were given; Table 1 shows these sample problems):

"1 [the arithmetic equation] would be a very familiar problem ... Same for 4 [the algebra equation] ... 3 [the arithmetic story] and 6 [the algebra story] add context ... Students would probably write 1 or 4 [equations] from any of the others before proceeding."

STORY PROBLEMS CAN BE EASIER

Some studies have identified circumstances where story problems are easier to solve than equations. Carraber, Carraber, & Schliemann (1987) found that

TABLE 1
Six Problem Categories Illustrating Two Difficulty Factors: Representation and Unknown-Position

Story Problem	Word Equation	Symbolic Equation
Result—unknown		
When Ted got home from his waiter job, he took the \$81.90 he earned that day and subtracted the \$66.00 he received in tips. Then he divided the remaining money by the 6 hr he worked and found his hourly wage. How much does Ted make per hour?	Starting with 81.9, if I subtract 66 and then divide by 6, I get a number. What is it?	Solve for x: $(81.90 - 66)/6 = x$
Start—unknown		
When Ted got home from his waiter job, he multiplied his hourly wage by the 6 hr he worked that day. Then he added the \$66.00 he made in tips and found he had earned \$81.90. How much does Ted make per hour?	Starting with some number, if I multiply it by 6 and then add 66, I get 81.9. What number did I start with?	Solve for x: $x \times 6 + 66 = 81.90$

Brazilian third graders were much more successful solving story problems (e.g., "Each pencil costs \$.03. I want 40 pencils. How much do I have to pay?") than solving matched problems presented symbolically (e.g., 3×40). Baranes, Perry, and Stigler (1989) used the same materials with U.S. third graders. U.S. children had higher overall success than the Brazilian children and, unlike the Brazilian children, did not perform better in general on story problems than on symbolic problems. However, Baranes and colleagues (1989) demonstrated specific conditions under which the U.S. children did perform better on story problems than on symbolic ones, namely, money contexts and numbers involving multiples of 25, corresponding to the familiar value of a quarter of a dollar.

If story problems are sometimes easier as the Carragher et al. (1987) and Baranes et al. (1989) results suggest, what is it about the story problem representation that can enhance student performance? Baranes and colleagues (1989) hypothesized that the situational context of story problems can make them easier than equivalent symbolic problems and suggested that the problem situation activates real-world knowledge ("culturally constituted systems of quantification," p. 316) that aids students in arriving at a correct solution.

Such an advantage of stories over symbolic forms can be explained within the solution phase of the problem-solving framework presented in Figure 1. Story problems can be easier when stories elicit different, more effective solution strategies than those elicited by equations. Past studies have demonstrated that different strategies can be elicited even by small variations in phrasing of the same story. For example, Hudson (1983) found that nursery school children were 17% correct on a standard story phrasing, "There are 5 birds and 3 worms. How many more birds are

8. After buying donuts at Wholly Donuts, Laura multiplies the number of donuts she bought by their price at \$0.22 per donut. Then she adds the \$0.22 change for the box they gave in and gets \$2.81. How many donuts did she buy?

$$.37x + .22 = 2.81$$

$$.37x = 2.59$$

$$x = 7$$

a. The normative strategy: Translate to algebra and solve algebraically

Starting with some number, if I multiply it by .37 and then add .22, I get 2.81. What number did I start with?

$$.37 \times 7 = 2.59$$

$$2.59 + .22 = 2.81$$

b. The guess-and-test strategy.

2. After hearing that Mom won a lottery prize, Bill took the amount she won and subtracted the \$64 that Mom kept for herself. Then he divided the remaining money among her 3 sons giving each \$26.50. How much did Mom win?

Mom won 179.50
 Mom kept 64.00
 143.50

179.50
 143.50
 36.00

c. The unwind strategy.

6. When Ted got home from his waiter job, he took the money he earned that day and multiplied it by his hourly wage. Then he added the \$66 that he made in tips and found he had earned \$81.90. How much did Ted earn that day?

$$(x - 66) \div 6 = 2.65$$

$$81.90 - 66 = 15.90$$

$$15.90 \div 6 = 2.65$$

$$x = 81.90$$

d. Translation to an algebra equation, which is then solved by the informal, unwind strategy.

7. After hearing that Mom won a lottery prize, Bill multiplied the \$26.50 she gave each son by 3 (because all 3 sons received the same amount). Then he added the \$64 that Mom kept for herself and found the total amount Mom won. How much did Mom win?

$$(26.5 \times 3) + 64 = x$$

$$79.50 + 64 = x$$

$$143.50 = x$$

e. A rare translation of a result-unknown story to an equation.

FIGURE 2 Examples of successful strategies used by students: (a) guess-and-test; (b) unwind; (c) translate to algebra and solve algebraically; (d) translate to algebra and solve by unwind; (e) translate to algebra and solve by arithmetic.

there than worms?" However, performance increased to an impressive 83% when the story is phrased as, "There are 5 birds and 3 worms. How many birds don't get a worm?" The latter phrasing elicits a match-and-count strategy that is more accessible for novice learners than the more sophisticated subtraction strategy elicited by the former, more standard phrasing.

The notion of a *situation model* (Kintsch & Greeno, 1985; Nathan et al., 1992) provides a theoretical account of how story problems described in one way can elicit different strategies than do equations or story problems described in other ways. In this account, problem solvers comprehend the text of a story problem by constructing a model-based representation of actors and actions in the story. Differences in the stories tend to produce differences in the situation models, which in turn can influence the selection and execution of alternative solution strategies. By this account, it is the differences in these strategies at the solution phase (see Figure 1) that ultimately accounts for differences in performance. For instance, Nines, Schliemann, and Carragher (1993) found that everyday problems were more likely to evoke oral solution strategies whereas symbolic problems evoked less effective written arithmetic strategies.

Process model developers of story problem solving (e.g., Bohrow, 1968; Cummins et al., 1988; Mayer, 1982b) have been careful to differentiate comprehension versus solution components of story problem solving. However, readers of the literature might be left with the impression that equation solving involves only a solution phase; in other words, that comprehension is not necessary. Although it is tempting to think of comprehension as restricted to the processing of natural language, clearly other external forms, such as equations, charts, and diagrams (cf. Larkin & Simon, 1987), must be understood or comprehended to be used effectively to facilitate reasoning. The lack of research on student comprehension of number sentences or equations may result from a belief that such processing is transparent or trivial for problem solvers at the algebra level. Regarding equations such as $(81.90 - 66)/6 = x$ and $x \times 6 + 66 = 81.90$ in Table 2, an algebra teacher commented that these could be solved "without thinking."

HYPOTHESES AND EXPERIMENTAL DESIGN

The two studies presented here are the first we know of that test the common belief that algebra learners have greater difficulty with story problems than matched equations.¹ Table 1 shows examples of the main factors manipulated in these studies. In addition to the main contrast between story problems and equations (first

¹Recent reviews of mathematics learning research (Kilpatrick, Swafford, & Findell, 2001), story problem research (Reed, 1998), and algebra research (Bednarz, Kieran, & Lee, 1996; Kieran, 1992) do not reference any such studies.

TABLE 2
Examples of the Four Cover Stories Used

Story Problem Cover Stories			
Donut	Lottery	Waiter	Basketball
After buying donuts at Whopley Donuts, Laura multiplies the 7 donuts she bought by their price of \$0.37 per donut. Then she adds the \$0.22 charge for the box they came in and gets the total amount she paid. How much did she pay?	After hearing that Mom won a lottery prize, Bill took the \$143.50 she won and subtracted the \$64.00 that Mom kept for herself. Then he divided the remaining money among her 3 sons giving each the same amount. How much did each son get?	When Ted got home from his waiter job, he multiplied his wage of \$2.65 per hour by the 6 hr he worked that day. Then he added the \$66.00 he made in tips and found how much he earned. How much did Ted earn that day?	After buying a basketball with his daughters, Mr. Jordan took the price of the ball, \$68.36, and subtracted the \$25 he contributed. Then he divided the rest by 4 to find out what each daughter paid. How much did each daughter pay?

and last columns in Table 1), we added an intermediate problem representation we refer to as *word problems* or *word equations* (middle column in Table 1). We included word equations to isolate effects of situational knowledge from effects of differences in language comprehension demands between verbal and symbolic forms. If stories cue useful situational knowledge, then we should find students making fewer errors on story problems than on both word and symbolic equations. If students' relevant symbolic comprehension skills lag behind their relevant verbal comprehension skills, then we should find them making more errors on symbolic equations than on either of the verbal forms.

Our goal was to explore representation effects for students at the transition from arithmetic to algebraic competence—a domain referred to as "early algebra" (cf. Carpenter & Levi, 1999; Kaput, in press). Thus, we included both relatively complex arithmetic problems (first row in Table 1) and relatively simple algebra problems (second row in Table 1). In the arithmetic problems, the problem unknown is the result of the process or sequence of operators described. These result-unknown problems are more complex than those used in prior research on elementary arithmetic problem solving (Briars & Larkin, 1984; Carpenter & Moser, 1984; Hiebert, 1982; Riley & Greeno, 1988). These problems involve two arithmetic operators (e.g., multiplication and addition) rather than one, decimals rather than only whole numbers, and more advanced symbolic notation such as parentheses and equations with a variable on the right side (e.g., $[81.90 - 66]/6 = x$). The relatively simple algebra problems we used are two-operator start-unknowns, that is, problems where the unknown is at the start of the arithmetic process described.

